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THE DERIVATIVE OF THE ENERGY FUNCTIONAL ALONG THE CRACK LENGTH IN PROBLEMS OF THE THEORY OF ELASTICITY†

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The problem of the equilibrium of a linear elastic body in a bounded two-dimensional of three-dimensional domain containing a cut (a crack) is considered. The boundary conditions on the sides of the crack have the form of inequalities and describe the condition for their mutual impenetrability. The derivative of the energy functional along the length of the crack is found and Griffith's formula is established. In the two-dimensional case, the Eshelby–Cherepanov–Rice integral is constructed along a curve enclosing the vertex of the crack and it is shown that it is independent of the integration path. An analogue of the Eshelby–Cherepanov–Rice integral is constructed for the three-dimensional case. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a linearly elastic body which, in the undeformed state, occupies a domain $D \subset R^p$ ($p = 2, 3$) and contains a crack. The crack is the modelled by a cut (when $p = 2$) and by a surface (when $p = 3$)

$$\Xi_{l+\delta} = \{(x_i) \mid \quad x_p = 0, \quad 0 < x_1 < l + \delta, \quad p = 3: -h < x_2 < h\}$$

where $h > 0, l > 0$, δ is a small parameter and the subscript i (and the subscript j below) take the values 1 and 2 when $p = 2$ and 1, 2 and 3 when $p = 3$ (Fig. 1). We assume that $\Xi_{l+\delta} \subset D$ for all sufficiently small δ and use the notation $\Omega_\delta = D \setminus \overline{\Xi}_{l+\delta}$, $\Omega = D \setminus \overline{\Xi}_l$.

The problem of the equilibrium of an elastic body containing a crack is formulated in the following way; it is required to find a vector function $W = (w^i)$ such that

$$-\sigma_{ij,j} = f_i \text{ in } \Omega; \quad f = (f_1, f_2) \in C^1(\overline{D}) \tag{1.1}$$

$$W = 0 \quad \text{on } \Gamma; \quad [W]n \geqslant 0 \quad \text{in } \Xi_l \tag{1.2}$$

Here, $\sigma_{ij} = \sigma_{ij}(W)$ are the components of the stress tensor, f_i are the components of the external load vector, and $[W] = W^+ - W^-$ is the discontinuity of the vector W on Ξ_l. The plus and minus superscripts correspond to positive and negative directions of the normal n to the line ($p = 2$) or to the surface ($p = 3$) $\Xi_{l+\delta}$. Summation is carried out over repeated indices.

We shall assume that Hooke's law is satisfied

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta^i_j \operatorname{div} W, \quad \varepsilon_{ij} = \varepsilon_{ij}(W) = (w^i_{,j} + w^j_{,i})/2 \tag{1.3}$$

where $\lambda \geqslant 0$ and $\mu > 0$ are Lamé parameters.

The formulation of problem (1.1), (1.2) is not complete. Actually, we consider the problem of minimizing the functional (henceforth the integration domain is not shown)

$$I(\Omega; W) = \tfrac{1}{2}\int \sigma_{ij}(W)\varepsilon_{ij}(W)d\Omega - \int fW d\Omega \tag{1.4}$$

in the convex, closed set

$$K_0 = \{(w^i) \in H^1(\Omega) \mid \quad w^i = 0 \quad \text{on } \Gamma, \quad [w^p] \geqslant 0 \quad \text{in } \Xi_l\} \tag{1.5}$$

In this case, the solution W of the minimization problem satisfies variational inequality (1.10) when

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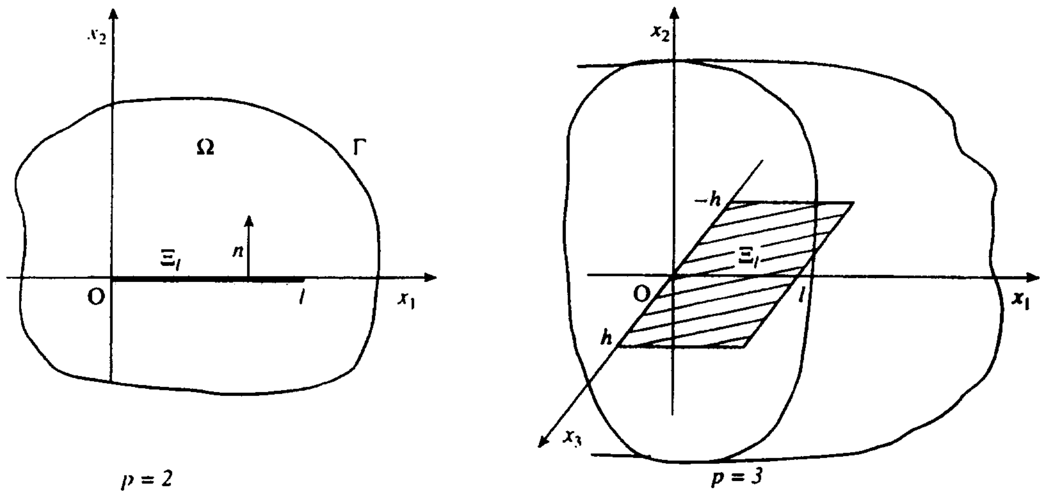

Fig. 1

$\delta = 0$ (see below). In particular, equilibrium equations (1.1) and boundary conditions (1.2) are satisfied. Other boundary conditions (see (2.14) when $p = 2$ and (3.3) when $p = 3$) will also be satisfied in the set Ξ_l in addition to the last condition of (1.2).

Together with problem (1.1), (1.2), we consider a family of perturbed problems, namely, it is required to find a vector function $W^\delta = (w^{i\delta})$ such that

$$-\sigma_{ij,j} = f_i \text{ in } \Omega_\delta; \quad \sigma_{ij} = \sigma_{ij}(W^\delta) \tag{1.6}$$

$$W^\delta = 0 \text{ on } \Gamma; \quad [W^\delta]n \geq 0 \quad \text{in} \quad \Xi_{l+\delta} \tag{1.7}$$

where σ_{ij} and ε_{ij} are related by Hooke's law (1.3).

As in the case of problem (1.1), (1.2), the formulation of problem (1.6), (1.7) is not complete. In reality, when speaking of problem (1.6), (1.7), we shall bear in mind the problem of minimizing the functional $I(\Omega_\delta; W)$ in the set K_δ. They differ from the corresponding functional (1.4) and the set (1.5) in that the domain Ω is replaced by Ω_δ an the set Ξ_l is replaced by $\Xi_{l+\delta}$.

The solution W^δ of the minimization problem satisfies the variational inequality (see inequality (1.10) below) and the second relation of (1.7) is part of the complete system of boundary conditions which are satisfied in the set $\Xi_{l+\delta}$.

The rate of change of the energy functional along the length of a crack is often involved in the formulation of fracture criteria [1]. Derivatives of the energy functional in the case of Poisson's equation and the line equations of the theory of elasticity (when $p = 2$) with classical boundary conditions of the form

$$\sigma_{22} = 0, \qquad \sigma_{12} = 0 \quad \text{in } \Xi_l \tag{1.8}$$

have been analysed (also, see [4–7]. In the case of the non-classical boundary conditions (1.2), the question of the derivatives of the energy functional has remained open up to the present time. The regularity of the solutions of elliptic boundary-value problems in non-smooth domains has been studied in [8–12] and other aspects of elliptic problems in domains with non-smooth boundaries have been considered in [11–17]. The dependence of the solutions of boundary-value problems on the shape of the domain has been investigated in [18]. Asymptotic expansions of the energy functional in the case of a perturbation of the dimensions of a crack have been constructed in [19–20].

The aim of this paper is to find the derivative of the energy functional along the length of a crack in the case of boundary conditions (1.2) in Ξ_l which ensure that the sides of the crack do not penetrate one another.

$$\lim_{\delta\to 0}[I(\Omega_\delta; W^\delta) - I(\Omega; W)]/\delta = dJ(\Omega_\delta)/d\delta\big|_{\delta=0} \quad (J(\Omega_\delta) = I(\Omega_\delta; W^\delta)) \tag{1.9}$$

Here, W^δ and W are the solutions of problems (1.6), (1.7) and (1.1), (1.2) respectively. Actually, W^δ is the solution of the following variational inequality

$$W^\delta \in K_\delta: \quad \int \sigma_{ij}(W^\delta)(\varepsilon_{ij}(V)-\varepsilon_{ij}(W^\delta))d\Omega_\delta \geqslant \int f(V-W^\delta)d\Omega_\delta \forall V \in K_\delta \tag{1.10}$$

and W is the solution of the analogous inequality when K_δ and Ω_δ are replaced by K_0 and Ω respectively.

In order to find the derivative (1.9), we carry out a one-to-one mapping of the domain Ω_δ onto Ω. We then make use of the variational properties of the solutions, which enables us to avoid the need to calculate the material derivative of W^δ.

2. THE TWO-DIMENSIONAL CASE

We select an arbitrary function $\theta \in C_0^\infty$ (D) which is such that $\theta = 1$ in a certain neighbourhood of the point $x_l = (l, 0)$. To simplify the formulae which follow below we will assume that $\theta = 0$ in the neighbourhood of the point (0, 0). We choose a transformation of the independent variables in the form

$$y_1 = x_1 - \delta\theta(x_1, x_2), \quad y_2 = x_2; \quad (y_1, y_2) \in \Omega, (x_1, x_2) \in \Omega_\delta \tag{2.1}$$

The Jacobian $q_\delta = 1 - \delta\theta_{x_l}$ of transformation (2.1) is positive for small δ.

Suppose $x = x(y, \delta)$ is the inverse transform of (2.1) and $W^\delta(x)$ is the solution of problem (1.6), (1.7). Then, $W^\delta(x) = W_\delta(y)\, y \in \Omega$. Suppose, also, that W is the solution of problem (1.1), (1.2). Then

$$\| W_\delta - W \|_{H^1(\Omega)} \to 0 \quad \text{when} \quad \delta \to 0 \tag{2.2}$$

We shall not present the proof of this convergence. It is analogous to the proof of Lemma 1 in [21] where Poisson's equation in a domain with a cut was considered.

Furthermore, it can be proved in the same way as described previously [21] that a constant $c > 0$ exists such that

$$\| W_\delta - W \|_{H^1(\Omega)} \leqslant c\delta \tag{2.3}$$

Using transformation (2.1), we obtain (henceforth, $i, j = 1, 2$ everywhere)

$$\int f_i w^\delta d\Omega_\delta = \int f_i^\delta w_\delta d\Omega, \quad w^\delta(x) = w_\delta(y), \quad f_i^\delta(y) = \frac{f_i(x(y,\delta))}{1-\delta\theta_{x_1}}$$

The derivatives

$$f_i'(y) = \lim_{\delta\to 0} \frac{f_i^\delta(y) - f_i^0(y)}{\delta} = \left.\frac{df_i^\delta}{d\delta}\right|_{\delta=0}$$

can now be found.

In fact, assuming y and δ are the independent variables in (2.1), we obtain $x = x(y, \delta)$. Differentiation of equalities (2.1) with respect to δ gives

$$\frac{dx_1}{d\delta} = \frac{\theta}{1-\delta\theta_{x_1}}, \quad \frac{dx_2}{d\delta} = 0$$

Hence,

$$\left.\frac{\partial f_i(x(y,\delta))}{\partial\delta}\right|_{\delta=0} = f_{ix_1}\left.\frac{dx_1}{d\delta}\right|_{\delta=0} + f_{ix_2}\left.\frac{dx_2}{d\delta}\right|_{\delta=0} = f_{iy_1}\theta \tag{2.4}$$

It follows from this that

$$\begin{aligned} f_i'(y) &= \lim_{\delta\to 0}\left(\frac{f_i(x(y,\delta))}{1-\delta\theta_{x_1}} - f_i(y)\right)\frac{1}{\delta} = \lim_{\delta\to 0}\frac{f_i(x(y,\delta)) - f_i(y)}{\delta} + \\ &+ \theta_{x_1} f_i(y)\big|_{\delta=0} = f_{iy_1}\theta + \theta_{y_1} f_i = \frac{\partial(\theta f_i)}{\partial y_1}(y) \end{aligned} \tag{2.5}$$

Moreover, taking account of the inclusion $f_i \in C^1(\overline{\Omega})$, we obtain when $\delta \to 0$

$$(f_i^\delta(y) - f_i^0(y))/\delta \to f_i'(y) \quad \text{in} \quad L^\infty(\Omega) \tag{2.6}$$

Now, suppose $W^\delta = (w^{i\delta})$ is the solution of problem (1.6), (1.7). We introduce the notation

$$w^{i\delta}(x) = w_\delta^i(y), \quad W^\delta(x) = W_\delta(y), \quad x \in \Omega_\delta, \quad y \in \Omega, \quad x = x(y, \delta)$$

The relations

$$\begin{aligned} w_{x_1}^{1\delta} &= w_{\delta y_1}^1(1 - \delta\theta_{x_1}), \quad w_{x_2}^{1\delta} = w_{\delta y_1}^1(-\delta\theta_{x_2}) + w_{\delta y_2}^1 \\ w_{x_1}^{2\delta} &= w_{\delta y_1}^2(1 - \delta\theta_{x_1}), \quad w_{x_2}^{2\delta} = w_{\delta y_1}^2(-\delta\theta_{x_2}) + w_{\delta y_2}^2 \end{aligned} \tag{2.7}$$

hold by virtue of transformation (2.1).

Since the equality

$$\sigma_{ij}(W^\delta)\varepsilon_{ij}(W^\delta) = (2\mu + \lambda)(\varepsilon_{11}^2(W^\delta) + \varepsilon_{22}^2(W^\delta)) + 2\lambda\varepsilon_{11}(W^\delta)\varepsilon_{22}(W^\delta) + 4\mu\varepsilon_{12}^2(W^\delta)$$

holds, then, by virtue of relations (2.7), it is possible to replace the integration domain Ω_δ by Ω in the formula for the energy functional, namely (henceforth $w_\delta^1 = u_\delta$, $w_\delta^2 = v_\delta$ everywhere).

$$\begin{aligned} &\frac{1}{2}\int \sigma_{ij}(W^\delta)\varepsilon_{ij}(W^\delta)d\Omega_\delta - \int fW^\delta d\Omega_\delta = \frac{1}{2}\int \frac{1}{q_\delta}(u_{\delta y_1}^2((2\mu + \lambda)q_\delta^2 - \mu\delta^2\theta_{x_2}^2) + \\ &+\mu u_{\delta y_2}^2 + v_{\delta y_1}^2((2\mu + \lambda)\delta^2\theta_{x_2}^2 + \mu q_\delta^2) + (2\mu + \lambda)v_{\delta y_2}^2 - 2\delta\mu u_{\delta y_1}u_{\delta y_2}\theta_{x_2} - \\ &-2\delta(\mu + \lambda)u_{\delta y_1}v_{\delta y_1}\theta_{x_2}q_\delta + 2\lambda u_{\delta y_1}v_{\delta y_2}q_\delta + 2\mu u_{\delta y_2}v_{\delta y_1}q_\delta - \\ &-2\delta(2\mu + \lambda)v_{\delta y_1}v_{\delta y_2}\theta_{x_2})d\Omega - \int f^\delta W_\delta d\Omega \end{aligned} \tag{2.8}$$

Formula (2.8), which gives the transformation of the energy functional, can be rewritten in the form

$$I(\Omega_\delta; W^\delta) = I_\delta(\Omega; W_\delta) \tag{2.9}$$

The inclusion $W_\delta \in K_0$ implies that $W^\delta \in K_\delta$ and, conversely, $W^\delta \in K_\delta$ implies that $W_\delta \in K_0$. This means that transformation (2.1) establishes a one-on-one correspondence between K_δ and K_0. In particular, what has been said means that

$$\min_{U \in K_\delta} I(\Omega_\delta; U) = \min_{U \in K_0} I_\delta(\Omega; U) \tag{2.10}$$

Then, by virtue of (2.9) and (2.10), we have

$$\begin{gathered} \Delta J/\delta = (I_\delta(\Omega; W_\delta) - I(\Omega; W))/\delta \leq \{I_\delta(\Omega; W)\}/\delta \\ (\Delta J = J(\Omega_\delta) - J(\Omega), \quad \{I_\delta(\Omega; W)\} = I_\delta(\Omega; W) - I(\Omega; W)) \end{gathered}$$

whence it follows that

$$\overline{\lim_{\delta \to 0}} \Delta J/\delta \leq \overline{\lim_{\delta \to 0}} \{I_\delta(\Omega; W)\}/\delta \tag{2.11}$$

On the other hand,

$$\underline{\lim_{\delta \to 0}} \Delta J/\delta \geq \underline{\lim_{\delta \to 0}} \{I_\delta(\Omega; W_\delta)\}/\delta \tag{2.12}$$

On taking relations (2.8), (2.6) and (2.2) into account, it can be shown that the right-hand sides of inequalities (2.11) and (2.12) are identical. This means that the derivative (1.9) exists. Direct calculation of the right-hand sides of inequalities (2.11) and (2.12) gives Griffith's formula

$$dJ(\Omega_\delta)/d\delta\big|_{\delta=0} = \tfrac{1}{2}\int (-(2\mu+\lambda)u_{y_1}^2\theta_{y_1} + \mu u_{y_2}^2\theta_{y_1} - 2\mu u_{y_1}u_{y_2}\theta_{y_2} - \mu v_{y_1}^2\theta_{y_1} + (2\mu+\lambda)\times$$
$$\times v_{y_2}^2\theta_{y_1} - 2(\mu+\lambda)u_{y_1}v_{y_1}\theta_{y_2} - 2(2\mu+\lambda)v_{y_1}v_{y_2}\theta_{y_2})d\Omega - \tag{2.13}$$
$$-\int (\theta f_1)_{y_1} u\,d\Omega - \int (\theta f_2)_{y_1} v\,d\Omega$$

This gives the value of the derivative of the energy functional along the path of the crack in the case of the two-dimensional theory of elasticity with non-linear boundary conditions (2.14) (see below) on the crack sides. We will first analyse the boundary conditions in Ξ_l in problem (1.1) and (1.2). According to results obtained previously [11], the solution W of problem (1.1), (1.2) satisfies the following boundary conditions

$$[v] \geqslant 0, \qquad \sigma_{22} \leqslant 0, \qquad [\sigma_{22}] = 0, \qquad \sigma_{12} = 0, \qquad \sigma_{22}[v] = 0 \quad \text{in} \quad \Xi_l \tag{2.14}$$

Moreover, it follows from results obtained previously in [8] that the solution of problem (1.1), (1.2) (that is, in fact of problem (1.10) when $\delta = 0$) has additional smoothness compared with the variational problem. In fact, for any $x \in \Xi_l$, a neighbourhood V of point x exists such that $W \in H^2(V\backslash\Xi_l)$. Consequently, according to the embedding theorems, the function W is continuous up to the crack sides and conditions (2.14) are satisfied almost everywhere in Ξ_l. Note that $\sigma_{22} = (2\mu + \lambda)v_{y2} + \lambda u_{y1}$. In addition to conditions (2.14), it can be proved that

$$[\sigma_{22}v_{y_1}] = \sigma_{22}[v_{y_1}] = 0 \text{ almost everywhere in } \Xi_l \tag{2.15}$$

Actually, by virtue of the continuity of the function v up to Ξ_l, the set

$$M = \{y \in \Xi_l \mid [v(y)] > 0\}$$

is open in Ξ_l. According to the last equality of (2.14), at any point $y \in M$, we have $\sigma_{22}(y) = 0$. It follows from this that $\sigma_{22}[v_{y1}] = 0$ almost everywhere in M. In the set $\Xi_l\backslash M$ we have $[v] = 0$. Consequently, $[v_{y1}] = 0$ (see [22], Chapter 2. Theorem A. 1) which also proves equality (2.15).

We will now prove that the right-hand side of equality (2.13) is independent of the choice of the function θ. It has already been established that the left-hand sides of inequalities (2.11) and (2.12) are identical. They are independent of θ, and hence the limit $\lim\Delta J/\delta$ when $\delta \to 0$ exists and is also independent of θ.

Additional properties of smoothness close to the point x_l can be established in some special cases. For example, suppose the solution W or problem (1.1), (1.2) possesses the property $[W] = 0$ in $B_{xl} \cap \Xi_l$, where B_{xl} is circle with centre at the point x_l. Then, using the method employed in [12], it can be proved that the equilibrium equations

$$-\sigma_{ij,j}(W) = f_i$$

are satisfied in the sense of distributions in B_{xl}. Consequently, $W \in H^3_{\text{loc}}(B_{xl})$. In addition to what has been said above, by virtue of the inclusion $f \in H^1(D)$, we obtain $W \in H^3_{\text{loc}}(\Omega)$. In this case, we shall have

$$dJ(\Omega(l))/dl = 0 \qquad (\Omega = \Omega(l)) \tag{2.16}$$

Actually, integration by parts on the right-hand side of (2.13) gives

$$\begin{aligned} dJ(\Omega(l))dl &= \int \theta((\sigma_{11,1} + \sigma_{12,2})u_{y_1} + (\sigma_{21,1} + \sigma_{22,2})v_{y_1})d\Omega(l) + \\ &+ \int \theta(f_1 u_{y_1} + f_2 v_{y_1})d\Omega(l) + \int \theta(\sigma_{22}[v_{y_1}] + [\sigma_{12}u_{y_1}])d\Xi_l \end{aligned} \tag{2.17}$$

According to relations (1.1), (2.14) and (2.15), the right-hand side of expression (2.17) is equal to zero, which also proves equality (2.16).

Note that the smoothness of the function $W \in H^2(B_{xl}\backslash\Xi_l)$ is sufficient for equality (2.16) to hold. In this case, the arguments presented above can be repeated and it can be proved that the right-hand side of expression (2.17) is equal to zero.

Griffith's formula (2.13) can be written in a form which does not contain the function θ. In order to do this, we select a circle $B_{xl}(r)$ of radius r and a boundary $\Gamma(r)$ such that $\theta = 1$ in $B_{xl}(r)$. Integration by parts in (2.13) then gives

$$dJ(\Omega(l))/dl = I_{\Gamma(r)} + \int (f_1 u_{y_1} + f_2 v_{y_1}) d(B_{x_l}(r) \setminus \Xi_l) \tag{2.18}$$

where

$$I_{\Gamma(r)} = \tfrac{1}{2}\int \nu_1((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2))d\Gamma(r) +$$
$$+\int \nu_2((2\mu+\lambda)v_{y_1} v_{y_2} + (\mu+\lambda)u_{y_1} v_{y_1} + \mu u_{y_1} u_{y_2})d\Gamma(r) \tag{2.19}$$

and (υ_1, υ_2) is the unit outward normal to $\Gamma(r)$.

Now suppose $f = 0$ in certain neighbourhood V of the point x_l. For fairly small r, we have $B_{xl}(r) \subset V$. The right-hand side of equality (2.18) is independent of r and the following property is therefore proved. Suppose W is the solution of problem (1.1), (1.2) and $f = 0$ in a certain neighbourhood of the point x_l. Then, the integral $I_{\Gamma(r)}$ is independent of r for all fairly small r. Moreover, the preceding arguments show that the integral I_C, which differs from $I_{\Gamma(r)}$ by the replacement of the circle $\Gamma(r)$ by an arbitrary curve C enclosing the point x_l, is independent of the curve C (Fig. 2). In this case, $\upsilon = (\upsilon_1, \upsilon_2)$ is a unit vector normal to curve C. Part of this curve may coincide with Ξ_l. Suppose $\Xi = \Xi_l \cap C$ Then, by virtue of relation (2.15), it is possible in I_C to integrate both along the side Ξ^+ as well as along the side Ξ^-.

We emphasize that the fact that the integral I_C is independent of the curve C holds in the case when $f = 0$ in a domain with a boundary C. An integral of the form I_C is called an Eshelby–Cherepanov–Rice integral. Note that the result that the integral is independent of the path has been obtained for the case of non-linear boundary conditions (2.14).

The well-known assertions that an Eshelby–Cherepanov–Rice integral is independent of the integration path refer to the case of boundary conditions (1.8) (see [23]). In this case, the integral is usually written in the form

$$\int (\sigma_{ij}\nu_j w^i_{y_1} - \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\nu_1) dC((w^1, w^2) = (u, v)) \tag{2.20}$$

It is clear that, in the case of boundary conditions (1.8), there is no need to integrate along $\Xi^{\pm}$, since the corresponding integrals are equal to zero. There is also no need to use equality (2.15). We will show by direct verification that the integrands in I_C and (2.20) are identical, so that the integral I_C has a classical form.

If the crack opening close to x_l in problem (1.1), (1.2) is non-zero, the boundary conditions close to x_l have the form $\sigma_{22} = \sigma_{12} = 0$, and we arrive at the classical case. We note here that the sufficient conditions which ensure that the crack sides do not come into contact have been given in [24].

Other invariant integrals along contours encompassing the crack tip also exist for the case of boundary conditions (1.8) [25].

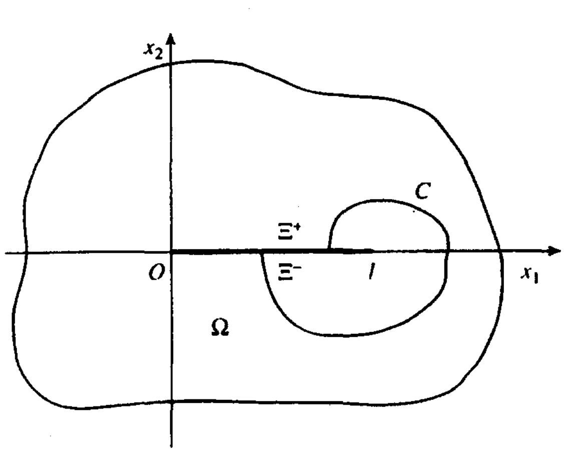

Fig. 2

3. THE THREE-DIMENSIONAL CASE

We now consider the function $\theta \in C_0^\infty(D)$, $\theta = 1$ in the neighbourhood of the set

$$L = \{(x_1, x_2, x_3) \mid x_1 = l,\ -h < x_2 < h,\ x_3 = 0\}$$

We assume that $\theta = 0$ in the neighbourhood of the set $L_0 = L|_{l=0}$. The transformation of the independent variables

$$y_1 = x_1 - \delta\theta(x_1, x_2, x_3),\quad y_2 = x_2,\quad y_3 = x_3 \tag{3.1}$$

in the case of small δ constitutes a one-to-one mapping of the domain Ω_δ onto Ω (x_1, x_2, x_3) Ω_δ $(y_1, y_2, y_3) \in \Omega$.

It is then possible to argue in the same manner as in Section 2 which gives Griffith's formula for the derivative of the energy functional ($w^1 = u$, $w^2 = v$, $w^3 = w$) everywhere henceforth)

$$\begin{aligned} dJ(\Omega_\delta)/d\delta\big|_{\delta=0} &= \tfrac{1}{2}\int ((2\mu+\lambda)\{-u_{y_1}^2\theta_{y_1} + v_{y_2}^2\theta_{y_1} + w_{y_3}^2\theta_{y_1} - 2v_{y_1}v_{y_2}\theta_{y_2} - \\ &- 2w_{y_1}w_{y_3}\theta_{y_3}\} + \mu\{u_{y_2}^2\theta_{y_1} - v_{y_1}^2\theta_{y_1} - w_{y_1}^2\theta_{y_1} + u_{y_3}^2\theta_{y_1} + v_{y_3}^2\theta_{y_1} + \\ &+ w_{y_2}^2\theta_{y_1} - 2u_{y_1}u_{y_3}\theta_{y_3} - 2u_{y_1}w_{y_1}\theta_{y_3} - 2v_{y_1}v_{y_3}\theta_{y_3} - 2v_{y_1}w_{y_2}\theta_{y_3} - 2v_{y_3}w_{y_1}\theta_{y_2} - \\ &- 2w_{y_1}w_{y_2}\theta_{y_2} - 2u_{y_1}u_{y_2}\theta_{y_2} - 2v_{y_1}u_{y_1}\theta_{y_2} + 2v_{y_3}w_{y_2}\theta_{y_1}\} + 2\lambda\{v_{y_2}w_{y_3}\theta_{y_1} - u_{y_1}w_{y_1}\theta_{y_3} - \\ &- v_{y_2}w_{y_1}\theta_{y_3} - v_{y_1}w_{y_3}\theta_{y_2} - u_{y_1}v_{y_1}\theta_{y_2}\})d\Omega - \int (\theta f_1)_{y_1} u\,d\Omega - \\ &- \int (\theta f_2)_{y_1} v\,d\Omega - \int (\theta f_3)_{y_1} w\,d\Omega \end{aligned} \tag{3.2}$$

It can be shown that the right-hand side of equality (3.2) is independent of θ.

It has been shown [11] that the solution W of problem (2.1), (2.2) satisfies the following boundary conditions

$$[w] \geq 0,\quad [\sigma_{33}] = 0,\quad \sigma_{33} \leq 0,\quad \sigma_{33}[w] = 0,\quad \sigma_{13} = 0,\quad \sigma_{23} = 0 \quad \text{на}\ \Xi_l \tag{3.3}$$

Moreover, it is shown that

$$\sigma_{33}[w_{,1}] = 0 \quad \text{almost everywhere in}\ \Xi_l \tag{3.4}$$

We will now write formula (3.2) in a form which does not contain the function θ. For this purpose, we consider a neighbourhood S_L of a set L with a smooth boundary Γ_L assuming that $\theta = 1$ in S_L. The unit outward normal to $(\upsilon_1, \upsilon_2, \upsilon_3)$ is denoted by Γ_L. Integrating by parts in (3.2), we obtain

$$\begin{aligned} dJ(\Omega_\delta)/d\delta\big|_{\delta=0} &= \int (f_1 u_{y_1} + f_2 v_{y_1} + f_3 w_{y_1})dS_L + \tfrac{1}{2}\int \nu_1((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2 - w_{y_3}^2) + \\ &+ \mu(v_{y_1}^2 + w_{y_1}^2 - u_{y_2}^2 - u_{y_3}^2 - v_{y_3}^2 - w_{y_2}^2 - 2v_{y_3}w_{y_2}) - 2\lambda v_{y_2}w_{y_3})d\Gamma_L + \\ &+ \int \nu_2((2\mu+\lambda)v_{y_1}v_{y_2} + \mu(u_{y_1}(u_{y_2} + v_{y_1}) + w_{y_1}(v_{y_3} + w_{y_2})) + \\ &+ \lambda(v_{y_1}(u_{y_1} + w_{y_3}))d\Gamma_L + \int \nu_3((2\mu+\lambda)w_{y_1}w_{y_3} + \\ &+ \mu(u_{y_1}(u_{y_3} + w_{y_1}) + v_{y_1}(v_{y_3} + w_{y_2})) + \lambda w_{y_1}(u_{y_1} + v_{y_2}))d\Gamma_L \end{aligned} \tag{3.5}$$

Denoting the functional defined by the right-hand side of Griffith's formula by $k(l, h, f)$, we obtain

$$J(\Omega_\delta) = J(\Omega) + k(l, h, f)\delta + o(\delta)$$

Note that $k(l, h, f)$ is independent of the choice of the neighbourhood S_L.

The method proposed for obtaining the derivative of the energy functional also enables us to treat more complex perturbations of the front of the crack in the three-dimensional case. Suppose, for example, that the front of the crack in the unperturbed state is defined by the equation $x_1 = g(x_2)$, where g is a specified function which satisfies the Lipshitz condition such that

$$g(-h) = g(h) = l$$

The perturbed front of the crack is described by the equation $x_1 = g(x_2) + \delta$. We assume that the transformation of the independent variables is identical to (3.1), where $\theta \in C_0^\infty(D)$ and $\theta = 1$ in the neighbourhood of the set

$$\{(x_1, x_2, x_3) \mid x_1 = g(x_2),\ -h < x_2 < h,\ x_3 = 0\}.$$

In this case Griffith's formula also has the form (3.2) with a domain Ω corresponding to the front $x_1 = g(x_2)$.

Similarly, suppose the front of the crack is defined by the equation $x_1 = l$ and the perturbed front has the form $x_1 = l + \delta g(x_2)$. The known function g is assumed to be fairly smooth and such that

$$g(-h) = g(h) = 0$$

We continue the function g outside of the interval $(-h, h)$ by zero and choose the function θ as in (3.1). Here, the transformation of the independent variables can be chosen in a form which differs from (3.1) by the introduction of the factor $g(x_2)$ in front of the function $\theta(x_1, x_2, x_3)$, and Griffith's formula has the form (3.2), in where the function θ is replaced by $g\theta$.

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